

Tunnelling from black holes and tunnelling into white holes

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Hawking radiation is nowadays being understood as tunnelling through black hole horizons. Here, the extension of the Hamilton-Jacobi approach to tunnelling for non-rotating and rotating black holes in different non-singular coordinate systems not only confirms this quantum emission from black holes but also reveals the new phenomenon of absorption into white holes by quantum mechanical tunnelling. The rôle of a boundary condition of total absorption or emission is also clarified.

1. A classical black hole has a horizon beyond which nothing escapes to the outside world. But there is a relation connecting the area of the horizon with the mass (and other parameters like the charge) indicating a close similarity [1] with the thermodynamical laws, suggesting the introduction of an entropy and a temperature [2]. This analogy was interpreted to be of quantum origin and the scale fixed after the (theoretical) discovery of radiation from black holes [3]. For the simplest Schwarzschild black hole, this radiation, which is thermal, has a temperature $T_H = \frac{\hbar}{4\pi r_h} = \frac{\hbar}{8\pi M}$, where r_h gives the location of the horizon in standard coordinates and M is the mass of the black hole. This result was derived by considering quantum massless particles in a Schwarzschild background geometry. A Hamilton - Jacobi equation based on a quantum mechanical tunnelling picture is more intuitive [4, 5, 6], but special care is needed to reproduce the temperature [7], avoiding incorrect values. Some other interesting tunnelling discussions may be found in [8].

A massless particle in a black hole background is described by the Klein-Gordon equation

$$\hbar^2 (-g)^{-1/2} \partial_\mu (g^{\mu\nu} (-g)^{1/2} \partial_\nu \phi) = 0. \quad (1)$$

One expands

$$\phi = \exp(-\frac{i}{\hbar} S + \dots) \quad (2)$$

to obtain to leading order in \hbar the Hamilton-Jacobi equation

$$g^{\mu\nu} \partial_\mu S \partial_\nu S = 0. \quad (3)$$

If the black hole horizon is a Killing horizon such that there is a Killing vector $\chi \equiv \frac{\partial}{\partial \tau}$ which is timelike in a vicinity of the horizon and becomes null on the horizon, a surface gravity κ can be defined as follows:

$$\chi \cdot \nabla \chi^\mu = \kappa \chi^\mu \quad \text{on the horizon.} \quad (4)$$

It is convenient to introduce a spatial coordinate λ by

$$\chi^2 = -2\kappa\lambda. \quad (5)$$

Then *near the horizon* the metric can be written in the form [9]

$$ds^2 = -2\kappa\lambda d\tau^2 + (2\kappa\lambda)^{-1} d\lambda^2 + \text{other terms.} \quad (6)$$

Separation of variables yields a solution independent of the other (angular) spatial coordinates:

$$S = E\tau + C + S_0(\lambda), \quad (7)$$

where E is to be interpreted as the energy and C is a constant. The equation for S_0 then reads

$$-\frac{E^2}{2\kappa\lambda} + (2\kappa\lambda) S_0'(\lambda)^2 = 0. \quad (8)$$

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The formal solution of this equation is

$$S_0(\lambda) = \pm E \int^\lambda \frac{d\lambda}{2\kappa\lambda}. \quad (9)$$

The sign ambiguity comes from the square root and corresponds to the fact that there can be incoming/outgoing solutions. There is, furthermore, a singularity at the horizon $\lambda = 0$, which has to be handled if one tries to find a solution across it. One way to skirt this pole is to change λ to $\lambda - i\epsilon$ in the denominator. This yields

$$\begin{aligned} S_{out} &= E\tau + C - \frac{E}{2\kappa} [i\pi + \int^\lambda d\lambda P(\frac{1}{\lambda})], \\ S_{in} &= E\tau + C + \frac{E}{2\kappa} [i\pi + \int^\lambda d\lambda P(\frac{1}{\lambda})], \end{aligned} \quad (10)$$

where $P()$ denotes the principal value. It was suggested in [7] that the imaginary part of C is to be determined so as to cancel the imaginary part of S_{in} and thus to ensure that the incoming probability is unity *in the classical limit* – when there is no reflection and everything is absorbed – instead of zero or infinity. Thus,

$$\begin{aligned} C &= -i\pi \frac{E}{2\kappa} + (Re\ C), \\ S_{out} &= E\tau - \frac{E}{2\kappa} [2i\pi + \int^\lambda d\lambda P(\frac{1}{\lambda})] + (Re\ C), \end{aligned} \quad (11)$$

implying a decay factor $\exp(-\pi E/\hbar\kappa)$ in the amplitude, and a factor $\exp(-2\pi E/\hbar\kappa)$ in the probability, in conformity with the standard value $\frac{\hbar\kappa}{2\pi}$ of the Hawking temperature. Note that in the Schwarzschild case, λ is $r - r_h$ and $\kappa = \frac{1}{2r_h}$.

However, the use of singular coordinates across horizons is not reliable, and the procedure must be regarded at best as an ad hoc one. Further, the introduction of an imaginary part in C may be felt to be undesirable in a non-singular framework. In the following pages, we shall go through a study of the non-rotating black hole as well as the Kerr black hole in several non-singular coordinate systems *without including an imaginary part in C* . An analysis of the results in the concluding section will show that not only can black holes radiate, even *white holes are capable of the classically forbidden process of absorption by quantum mechanical tunnelling*. The need for an imaginary part of C in the case of singular coordinate systems will also become clear.

2. Let us first consider the advanced Eddington-Finkelstein coordinates to describe the Schwarzschild black hole. Here one employs the coordinate v instead of t , where

$$dv = dt + \frac{dr}{1 - \frac{r_h}{r}}. \quad (12)$$

The metric reads

$$ds^2 = -(1 - \frac{r_h}{r})dv^2 + 2dvdr + r^2 d\Omega^2. \quad (13)$$

The Hamilton-Jacobi equation for spherically symmetric S is

$$2 \frac{\partial S}{\partial v} \frac{\partial S}{\partial r} + (1 - \frac{r_h}{r}) (\frac{\partial S}{\partial r})^2 = 0, \quad (14)$$

in which we can put $\frac{\partial S}{\partial v} = E$ to obtain two solutions

$$S_{out} = Ev - 2E \int \frac{dr}{1 - \frac{r_h}{r}}, \quad S_{in} = Ev. \quad (15)$$

If we treat the singularity in S_{out} as above, we find an imaginary part $-2\pi r_h E$, which leads to the correct Hawking temperature, and the incoming solution involves no decay or amplification. The reason why the imaginary parts here differ from those with Schwarzschild coordinates is that the transformation between these coordinates involves the singular r integral.

It is interesting to repeat the calculation with retarded Eddington-Finkelstein coordinates. Here one uses

$$du = dt - \frac{dr}{1 - \frac{r_h}{r}} \quad (16)$$

instead of dv and the metric reads

$$ds^2 = -(1 - \frac{r_h}{r})du^2 - 2dudr + r^2d\Omega^2. \quad (17)$$

The Hamilton-Jacobi equation for spherically symmetric S is

$$-2\frac{\partial S}{\partial u}\frac{\partial S}{\partial r} + (1 - \frac{r_h}{r})(\frac{\partial S}{\partial r})^2 = 0, \quad (18)$$

so that

$$S_{out} = Eu, \quad S_{in} = Eu + 2E \int \frac{dr}{1 - \frac{r_h}{r}}. \quad (19)$$

Here, S_{out} has no imaginary part but S_{in} has an imaginary part $2r_h\pi E$, which means that outward emission is complete, but *absorption is inhibited* by a factor $\exp(-2\pi r_h E/\hbar)$ in the amplitude, and a factor $\exp(-4\pi r_h E/\hbar)$ in the probability. We postpone the interpretation of this situation to the last section.

We now come to non-singular coordinates due to Painlevé, which were mentioned in [7]. There are two metrics of this kind:

$$ds^2 = -(1 - \frac{r_h}{r})dt^2 \pm 2\sqrt{\frac{r_h}{r}}dt dr + dr^2 + r^2d\Omega^2. \quad (20)$$

Only the upper sign in the cross term was used in the discussion given in [7]. We shall extend the discussion to the lower sign and find a difference. The Hamilton-Jacobi equation for spherically symmetric S is

$$-(\frac{\partial S}{\partial t})^2 \pm 2\sqrt{\frac{r_h}{r}}\frac{\partial S}{\partial t}\frac{\partial S}{\partial r} + (1 - \frac{r_h}{r})(\frac{\partial S}{\partial r})^2 = 0. \quad (21)$$

The r -dependent part of S is then

$$S_0(r) = E \int^r \frac{dr(\mp\sqrt{\frac{r_h}{r}} + (\pm 1))}{1 - \frac{r_h}{r}}, \quad (22)$$

where the \mp sign comes from the \pm in the above equation, while the (± 1) appears when the quadratic equation is solved. Thus, as in the coordinates studied above,

$$\begin{aligned} S_{out} &= Et - E[r_h \cdot i\pi(1 \pm 1) + \\ &\quad \int^r dr P(\frac{1}{r - r_h})r(1 \pm \sqrt{\frac{r_h}{r}})], \\ S_{in} &= Et + E[r_h \cdot i\pi(1 \mp 1) + \\ &\quad \int^r dr P(\frac{1}{r - r_h})(1 \mp \sqrt{\frac{r_h}{r}})], \end{aligned} \quad (23)$$

For the upper metric, $Im S_{out} = -2Er_h\pi$, reproducing the Hawking temperature, while absorption is complete. When the lower sign is taken in the metric, emission is complete, but absorption is inhibited by the factor $\exp(-2\pi r_h E/\hbar)$ in the amplitude, and the factor $\exp(-4\pi r_h E/\hbar)$ in the probability.

Let us next look at Lemaître coordinates for the same black hole – cf. [4]. Here, the metrics are (again one can choose either the + or the - sign)

$$ds^2 = -d\tau^2 + \frac{dR^2}{[\frac{3}{2r_h}(R \mp \tau)]^{2/3}} + r_h^2[\frac{3}{2r_h}(R \mp \tau)]^{4/3}d\Omega^2. \quad (24)$$

The Hamilton-Jacobi equation for spherically symmetric S is

$$-(\frac{\partial S}{\partial \tau})^2 + [\frac{3}{2r_h}(R \mp \tau)]^{2/3}(\frac{\partial S}{\partial R})^2 = 0. \quad (25)$$

We may write

$$S = S_-(R - \tau) + S_+(R + \tau). \quad (26)$$

For the upper sign, $S'_+ = E/2$, where the normalization of the energy is fixed by noting that $\tau = \frac{1}{2}(R + \tau) - \frac{1}{2}(R - \tau)$ and in the asymptotic region the dR^2 piece of the metric drops out. For the lower sign, the situation gets reversed: $S'_- = -E/2$. The Hamilton-Jacobi equation thus reduces to

$$(\pm E/2 + S'_\mp)^2 [\frac{3}{2r_h}(R \mp \tau)]^{2/3} = (E/2 \mp S'_\mp)^2, \quad (27)$$

so that

$$S = \pm \frac{E}{2}(R \pm \tau) + \frac{E}{2} \left(\frac{2r_h}{3} \right) \int d\xi_\mp^3 \frac{(\pm 1) \mp \xi_\mp}{\xi_\mp \pm (\pm 1)}, \quad (28)$$

where $\xi_\mp \equiv [\frac{3}{2r_h}(R \mp \tau)]^{1/3}$. The terms (± 1) in the ξ integral are unrelated to the choice of sign in the metric and arise on solving the quadratic equation for S_\pm .

The upper sign in (± 1) corresponds to incoming particles and the lower sign corresponds to outgoing ones. For the upper sign in the metric, there is no singularity in the incoming case, and no imaginary part in the integral, while there is a singularity at the horizon in the outgoing case, where the imaginary part of the integral comes out to be $-2\pi E r_h$. For the lower sign in the metric, there is no singularity in the outgoing case, and no imaginary part in the integral, while there is a singularity at the horizon $\xi = 1$ in the incoming case where the imaginary part of the integral comes out to be $2\pi E r_h$. These observations lead to the expected decay factor $\exp(-2\pi E r_h/\hbar)$ in the amplitude and $\exp(-4\pi E r_h/\hbar)$ in the probability for emission with the upper metric, where absorption is complete. The same factors hold also for absorption with the lower metric, where emission is complete.

It may be instructive to note the differences of this treatment with that in [4]. The normalization of E is different. Only one metric is used at a time, not both simultaneously. Lastly, it is not necessary to invoke global factors to derive the Hawking temperature for the upper sign of the metric.

One can also discuss the black hole in Kruskal coordinates. The metric is written as

$$ds^2 = -\frac{4r_h^3}{r} e^{-\frac{r}{r_h}} dU dV + r^2 d\Omega^2, \quad (29)$$

where r is related to the standard Kruskal coordinates by

$$UV = (1 - \frac{r}{r_h}) e^{\frac{r}{r_h}}. \quad (30)$$

The Hamilton-Jacobi equation satisfied by a spherically symmetric S is

$$\frac{\partial S}{\partial U} \frac{\partial S}{\partial V} = 0. \quad (31)$$

There are two possibilities: S is dependent on only U (outgoing) or only V (incoming). These solutions take the form

$$S_{out} = -\frac{E}{\kappa} \int \frac{dU}{U}, \quad S_{in} = \frac{E}{\kappa} \int \frac{dV}{V}. \quad (32)$$

An imaginary piece arises in the U integral, and as in the above cases, it is $i\pi$. It should be pointed out here that U changes sign across the horizon, unlike proper radial variables, which become imaginary on one side; thus an $i\pi$ factor arises for U , just as for r , though it gets halved for proper radial variables. This $i\pi$ is just what is needed to reproduce the Hawking temperature $\frac{\hbar\kappa}{2\pi}$ for emission. Note that S_{in} has a non-zero imaginary part $\frac{E\pi}{\kappa}$, so that absorption is inhibited by the factor $\exp(-2\pi E r_h/\hbar)$ in the amplitude and $\exp(-4\pi E r_h/\hbar)$ in the probability.

Lastly we consider the Kerr black hole, which is a rotating one. It can be described by the metric

$$ds^2 = -F(r, \theta) dt^2 + \frac{dr^2}{g(r, \theta)} + K(r, \theta) [d\phi - \frac{H(r, \theta)}{K(r, \theta)} dt]^2 + \Sigma(r, \theta) d\theta^2, \quad (33)$$

where

$$\begin{aligned} F(r, \theta) &= \frac{\Delta(r)\Sigma(r, \theta)}{(r^2 + a^2)^2 - \Delta(r)a^2 \sin^2 \theta}, \\ g(r, \theta) &= \frac{\Delta(r)}{\Sigma(r, \theta)}, \quad H(r, \theta) = \frac{2aMr \sin^2 \theta}{\Sigma(r, \theta)}, \\ \Sigma(r, \theta) &= r^2 + a^2 \cos^2 \theta, \quad \Delta(r) = r^2 + a^2 - 2Mr, \\ K(r, \theta) &= \frac{(r^2 + a^2)^2 - \Delta(r)a^2 \sin^2 \theta}{\Sigma(r, \theta)} \sin^2 \theta. \end{aligned} \quad (34)$$

Here M represents the mass and a is a rotation parameter. The Hamilton-Jacobi equation for $S(t, r, \phi)$ at a fixed $\theta = \theta_0$ near the horizon, which is located at $r_+ = M + \sqrt{M^2 - a^2}$, satisfying $\Delta(r_+) = 0$, is

$$g(r, \theta_0) \left(\frac{\partial S}{\partial r} \right)^2 - \frac{1}{F(r, \theta_0)} \left(\frac{\partial S}{\partial t} + \Omega \frac{\partial S}{\partial \phi} \right)^2 = 0, \quad (35)$$

where $\Omega \equiv \frac{H(r_+, \theta)}{K(r_+, \theta)} = \frac{a}{r_+^2 + a^2}$. Both $F(r, \theta_0)$ and $g(r, \theta_0)$ have simple zeroes at the horizon. Near the horizon, $\sqrt{Fg} = \frac{\Delta}{r_+^2 + a^2}$, which is independent of θ_0 . Discussions of rotating black holes are available in [5, 6]. We shall however consider advanced Eddington-Finkelstein coordinates defined by

$$dv = dt + dr \frac{r^2 + a^2}{\Delta}, \quad d\psi = d\phi + dr \frac{a}{\Delta}. \quad (36)$$

Then it is straightforward to see that near the horizon, $S(v, r, \psi)$ satisfies

$$\begin{aligned} \left(\frac{\partial S}{\partial r} + \frac{\partial S}{\partial v} \frac{r_+^2 + a^2}{\Delta} + \frac{\partial S}{\partial \psi} \frac{a}{\Delta} \right)^2 &= \left(\frac{r_+^2 + a^2}{\Delta} \right)^2 \left(\frac{\partial S}{\partial v} \right. \\ &\quad \left. + \Omega \frac{\partial S}{\partial \psi} \right)^2. \end{aligned} \quad (37)$$

There are two solutions:

$$\begin{aligned} S_{out} &= Ev + J\psi - 2(E + \Omega J) \int \frac{dr(r_+^2 + a^2)}{\Delta}, \\ S_{in} &= Ev + J\psi. \end{aligned} \quad (38)$$

The incoming solution shows that absorption is complete, while the outgoing solution has an imaginary part and involves a decay factor of $\exp[-(E + \Omega J)\pi(r_+^2 + a^2)/\hbar(r_+ - M)]$ in the amplitude and hence $\exp[-2(E + \Omega J)\pi(r_+^2 + a^2)/\hbar(r_+ - M)]$ in the probability, indicating the known temperature $\frac{\hbar(r_+ - M)}{2\pi(r_+^2 + a^2)}$.

The calculation may be repeated with retarded Eddington-Finkelstein coordinates defined by

$$du = dt - dr \frac{r^2 + a^2}{\Delta}, \quad d\chi = d\phi - dr \frac{a}{\Delta}, \quad (39)$$

leading to

$$\begin{aligned} S_{out} &= Eu + J\chi, \\ S_{in} &= Eu + J\chi + 2(E + \Omega J) \int \frac{dr(r_+^2 + a^2)}{\Delta}. \end{aligned} \quad (40)$$

In this case the outgoing solution has no imaginary part and shows complete emission while the incoming solution has an imaginary part leading to the same decay factor as above.

3. In this letter we have sought to analyse tunnelling across horizons in the Hamilton-Jacobi approach using non-singular coordinates and forsaking the imaginary part of the integration constant C that was found to be necessary in other coordinates [7]. In the preceding sections we have observed that different non-singular coordinate systems lead to two kinds of results. For one set, the incoming solutions do not contain any imaginary parts, implying complete absorption. Correspondingly, the outgoing solutions contain an imaginary part which is just right to support the interpretation of the inhibition associated with the Hawking temperature. This is comforting. The advanced Eddington-Finkelstein coordinates, the upper Painlevé and the upper Lemaître coordinates belong to this category.

For another set of non-singular coordinates, the situation is just the reverse. The outgoing solution has no imaginary part, suggesting that emission is uninhibited, while the incoming solution has just the right imaginary part to suggest an inhibition of the kind which would be expected with the Hawking temperature. The retarded Eddington-Finkelstein coordinates, the lower Painlevé and the lower Lemaître coordinates belong to this category. There is also the Kruskal choice of coordinates where both the outgoing and the incoming solutions contain imaginary parts, corresponding to the kind of inhibition seen above. What can one make of this?

One has to remember that while the advanced Eddington-Finkelstein coordinates, the upper Painlevé and the upper Lemaître coordinates all describe black holes with their exterior and interior regions, the retarded Eddington-Finkelstein coordinates, the lower Painlevé and the lower Lemaître coordinates actually describe their time-reversed

versions, *i.e.*, the exterior regions with white hole interiors. Just as black holes may absorb matter classically but can emit matter only quantum mechanically and with a specific temperature, white holes may emit matter classically but can absorb matter only quantum mechanically and again with a specific temperature. This is the meaning of the imaginary parts found above. The Kruskal case is different: here one simultaneously has a black hole region and a white hole region. However, if we consider the black hole and the exterior regions only, which is equivalent to restricting the V -coordinate to the domain $0 \leq V \leq \infty$, we effectively recover the results obtained in the advanced Eddington-Finkelstein coordinates. Similar conclusions hold when we consider the white hole and the exterior regions only.

White holes are perhaps less familiar objects than black holes, but they are essentially time-reversed versions of the latter. Hawking has argued [10] that while black holes may be created by classical collapse, their disintegration is a quantum phenomenon. By time-reversal, the creation of white holes is a quantum phenomenon, but their disintegration is classical. This is consistent with our results, though of course we consider only eternal black or white holes, which are used only as providing gravitational backgrounds.

As regards the integration constant C , we have dispensed with it in discussions using non-singular coordinates, but it is possible for an imaginary part of C to appear for Schwarzschild and other coordinates which are not well-defined across the horizon. Distinct boundary conditions (imaginary part of C) are to be imposed to fix whether a black or a white hole is being considered. Only by insisting that absorption is uninhibited can we be sure of getting a black hole and the correct emission probability. Similarly, a boundary condition of unhindered emission will ensure the identification of a white hole and its absorption probability. These boundary conditions can be tuned for singular coordinates by allowing C to have an imaginary part, which corresponds to an imaginary part in the transformation taking a non-singular system of coordinates to another system. Similar boundary conditions for non-singular coordinates lead to $\text{Im } C = 0$ as is implicit in our calculations.

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